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On a generalization of absolute neighborhood retracts

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ABSTRACT

In this paper we generalize the concept of absolute neighborhood retract by introducing the notion of absolute neighborhood multi-retract. Furthermore, the Lefschetz fixed point theorem for admissible maps defined on absolute neighborhood multi-retracts is proved.

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1. Introduction

The main aim of this paper is to generalize the notions of absolute retract (AR) and absolute neighborhood retract (ANR). We introduce and study two new classes of spaces, called absolute multi-retracts and absolute neighborhood multi-retracts (written AMR and ANMR, respectively). In addition, we shall prove that several properties of ARs and ANRs are also valid for our spaces. The class of all absolute neighborhood multi-retracts is quite large; it contains in particular all absolute neighborhood retracts and some metric spaces which are not movable. Using the theory of shape, we are going to show that an ANMR need not be an ANR. Moreover, we shall show that an ANMR need not be an approximative absolute neighborhood retract in the sense of Clapp (see Definition 2.6 below).

This paper is organized as follows. It consists of five sections. After this Introduction, Section 2 is devoted to some preliminaries. In Section 3 we study the so-called cell-like maps. In particular, we shall construct some examples of cell-like maps from a noncontractible ANR onto a space which is not an ANR. In Section 4 we introduce two new classes of spaces which we will call absolute multi-retracts and absolute neighborhood multi-retracts, respectively. Next, we shall present some basic properties of such spaces. In particular, using cell-like maps, we show that the class of absolute neighborhood multi-retracts is essentially larger than that of the class of absolute neighborhood retracts. In Section 5, by using the homological techniques developed in [10,11,13,15,16], we prove that the fixed point theory can be extended to ANMRs.

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2. Preliminaries

Let us describe briefly our notation and basic terminology. In this paper all topological spaces are assumed to be metric and all single-valued maps are assumed to be continuous. Given a metric space (X, d) , by $\text{int } A$, \bar{A} and ∂A we denote the interior, closure and boundary of a set A in X . If $x \in X$ and $\varepsilon > 0$, then we put

$$B(x, \varepsilon) := \{y \in X \mid d(x, y) < \varepsilon\}, \quad D(x, \varepsilon) := \{y \in X \mid d(x, y) \leq \varepsilon\}.$$

A continuous surjection $f : X \rightarrow Y$ is called proper provided for every compact set $B \subset Y$ the counter image $f^{-1}(B)$ is also compact.

Definition 2.1. A map $p : (X, X_0) \rightarrow (Y, Y_0)$ of pairs is said to be a Vietoris map provided the following conditions are satisfied:

- (i) $p : X \rightarrow Y$ is proper,
- (ii) $p^{-1}(Y_0) = X_0$,
- (iii) the set $p^{-1}(y)$ is acyclic for every $y \in Y$.¹

The importance of the above definition is expressed in the following theorem.

Theorem 2.1. (See [11].) If $f : (X, X_0) \rightarrow (Y, Y_0)$ is a Vietoris map, then the induced map $f_* : \check{H}_*(X, X_0; \mathbb{Q}) \rightarrow \check{H}_*(Y, Y_0; \mathbb{Q})$ is a linear isomorphism.

In this paper, by a multivalued map $\varphi : X \multimap Y$ we mean a function $X \rightarrow 2^Y$ such that, for each $x \in X$, $\varphi(x)$ is non-empty and compact. For a multivalued map $\varphi : X \multimap Y$ and any subset $U \subset Y$ we put:

$$\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \subset U\}.$$

A multivalued map $\varphi : X \multimap Y$ is said to be upper semicontinuous (u.s.c.) if, for every open subset U of Y , the set $\varphi^{-1}(U)$ is open in X . A multivalued map $\varphi : X \multimap Y$ is called compact provided the image $\varphi(X)$ of X under φ is contained in a compact subset of Y . Throughout the paper, all multivalued maps will be assumed to be upper semicontinuous. An upper semicontinuous map with acyclic values (in the sense of the Čech homology considered here) will be called an acyclic map. Given two multivalued maps $\varphi_1 : X_1 \multimap Y_1$ and $\varphi_2 : X_2 \multimap Y_2$, we can define the product map $\varphi_1 \times \varphi_2 : X_1 \times X_2 \multimap Y_1 \times Y_2$ as follows $(\varphi_1 \times \varphi_2)(x_1, x_2) := \varphi_1(x_1) \times \varphi_2(x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$.

Remark 2.1. Note that the basic terminology concerning multivalued maps used throughout the paper is taken from the book by L. Górniewicz [13].

Observe that any multivalued map $\varphi : X \multimap Y$ admits the standard factorization through the graph $\Gamma_\varphi := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$, i.e. there exists a diagram

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y$$

where $p_\varphi : \Gamma_\varphi \rightarrow X$ and $q_\varphi : \Gamma_\varphi \rightarrow Y$ are the projections such that $\varphi(x) = q_\varphi(p_\varphi^{-1}(x))$ for all $x \in X$. Consequently, this suggests the following definition.

Definition 2.2. (See [11].) A multivalued map $\varphi : X \multimap Y$ is called admissible provided that there exists a metric space Γ and two continuous maps $p : \Gamma \rightarrow X$ and $q : \Gamma \rightarrow Y$ such that the following conditions are satisfied:

- (i) p is a Vietoris map,
- (ii) $\varphi(x) = q(p^{-1}(x))$ for any $x \in X$.

In what follows, the pair (p, q) determining an admissible map φ will be called a selected pair for φ (written $(p, q) \subset \varphi$). Now we shall list two basic properties of admissible maps.

Theorem 2.2. (See [11].) Let $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$ be two admissible maps. Then the composition $\psi \circ \varphi : X \multimap Z$ is an admissible map.

¹ In this paper by \check{H}_* we shall denote the Čech homology functor with compact carriers and coefficients in the field of rational numbers \mathbb{Q} (see [2] and [13]). A non-empty space X is called acyclic if $\check{H}_*(X; \mathbb{Q}) = \check{H}_*(pt; \mathbb{Q})$, where pt stands for a one-point space.

Lemma 2.1. (See [11].) If $\varphi: X \multimap Y$ is an admissible map, $Y_0 \subset Y$ and $X_0 = \varphi^{-1}(Y_0)$, then the restriction $\varphi_0: X_0 \multimap Y_0$ of φ to the pair (X_0, Y_0) is an admissible map.

Lemma 2.2. (See [12].) Let $\varphi_1: X_1 \multimap Y_1$ and $\varphi_2: X_2 \multimap Y_2$ be two admissible maps. Then the product map $\varphi_1 \times \varphi_2: X_1 \times X_2 \multimap Y_1 \times Y_2$ is also admissible.

It is easy to see that if an upper semicontinuous multivalued map φ has acyclic values, then φ is admissible (see [13]).

Now, following K. Borsuk (see [4]) we recall the notion of absolute retract and the notion of absolute neighborhood retract. For this purpose it is useful to use the notion of an r -map.

Definition 2.3. A map $r: X \rightarrow Y$ of a space X onto a space Y is said to be an r -map if there is a map $s: Y \rightarrow X$ such that $r \circ s = id_Y$.

Definition 2.4. A metric space X is called an absolute neighborhood retract (notation: $X \in ANR$) provided there exists an open subset U of some normed linear space E and an r -map $r: U \rightarrow X$ from U onto X .

Definition 2.5. A metric space X is called an absolute retract (notation: $X \in AR$) provided there exists a normed linear space E and an r -map $r: E \rightarrow X$ from E onto X .

Some important properties of ARs and of ANRs are established in the following:

Proposition 2.1. (See [4].) Suppose that the metrizable space X is the union of two closed subsets X_1 and X_2 and let $X_0 = X_1 \cap X_2$. Then:

- (i) if $X_0, X_1, X_2 \in AR$, then $X \in AR$;
- (ii) if $X_0, X_1, X_2 \in ANR$, then $X \in ANR$.

One can prove that if X is an ANR, then any open subset of X is also an ANR (see [4]).

Now we shall recall a generalization of the concept of absolute neighborhood retract, which was introduced by M.H. Clapp (see [7]).

Definition 2.6. We shall say that a compact metric space X is approximative absolute neighborhood retract in the sense of Clapp (notation: $X \in AANR_C$) provided for every $\varepsilon > 0$ there exists an open subset U of some normed linear space E and two maps $r_\varepsilon: U \rightarrow X$, $s_\varepsilon: X \rightarrow U$ such that $d(x, r_\varepsilon(s_\varepsilon(x))) < \varepsilon$ for any $x \in X$ (a function r_ε is called an ε -retraction).

It is easy to see that every compact ANR is an $AANR_C$. This follows from the fact that an r -map is an ε -retraction for every positive number ε . In addition, Example 5.1, given in Section 5, shows that an $AANR_C$ need not be an ANR.

In this paper, we shall use the following result known as Schauder's approximation theorem (cf. [15]).

Theorem 2.3. Let X be a metric space and let U be an open subset of a normed linear space $(E, \|\cdot\|)$. In addition, let $i: X \rightarrow U$ be a compact map. Then for each sufficiently small $\varepsilon > 0$ there exists a finite polyhedron $K_\varepsilon \subset U$ and a map $i_\varepsilon: X \rightarrow U$ such that:

- (i) $\|x - i_\varepsilon(x)\| < \varepsilon$ for all $x \in X$,
- (ii) $i_\varepsilon(X) \subset K_\varepsilon$,
- (iii) the maps $i, i_\varepsilon: X \rightarrow U$ are homotopic.

In what follows, by $X \sqcup Y$ we shall denote the disjoint union of two metric spaces X and Y (recall that $X \sqcup Y$ is metrizable if and only if X and Y are metrizable—see [9]).

Definition 2.7. Let X and Y be two compact metric spaces and let A be a closed subset of X . Let $f: A \rightarrow Y$ be a continuous map. Then we denote by $X \sqcup_f Y$ the quotient space $X \sqcup Y / \sim$, where \sim is the equivalence relation on $X \sqcup Y$ defined as follows: $u \sim v$ if one of the following is true:

- (1) $u = v$;
- (2) $u, v \in A$ and $f(u) = f(v)$;
- (3) $u \in A$ and $v = f(u) \in Y$.

Remark 2.2. In general, quotient spaces of metrizable spaces are not necessarily metrizable. However, the quotient space $X \sqcup_f Y$ defined above is metrizable. For more details about metrization theorems we refer the reader to [9].

In this paper, we shall make use of the following evident fact.

Proposition 2.2. (See [23].) *Let X and Y be two compact metric spaces and let A be a closed subset of X . Let $f : A \rightarrow Y$ be a continuous map. Then the composite $X \hookrightarrow X \sqcup Y \xrightarrow{\pi} X \sqcup_f Y$ maps $X \setminus A$ homeomorphically onto an open subset of $X \sqcup_f Y$, where π is the natural quotient mapping. In addition, the composite $Y \hookrightarrow X \sqcup Y \xrightarrow{\pi} X \sqcup_f Y$ is a homeomorphism from Y to a subspace of $X \sqcup_f Y$.*

For the remainder of this section we present some definitions and results that will be needed to prove the Lefschetz fixed point theorem (see the last section of this paper). We begin by recalling the notion of the Leray trace and the generalized Lefschetz number. For this purpose we introduce some terminology to simplify the exposition of basic facts on the Leray trace. In what follows, all the vector spaces are taken over \mathbb{Q} . Let $L : E \rightarrow E$ be an endomorphism. Let us put

$$N(L) = \{x \in E \mid L^{(n)}(x) = 0 \text{ for some } n\},$$

where $L^{(n)}$ is the n th iterate of L . Since $L(N(L)) \subset N(L)$, we have the induced endomorphism $\tilde{L} : \tilde{E} \rightarrow \tilde{E}$ defined by the formula $\tilde{L}([x]) = [L(x)]$, where $\tilde{E} = E/N(L)$ is the factor space and $[x]$ stands for the equivalence class of $x \in E$. An endomorphism $L : E \rightarrow E$ is called admissible if $\dim \tilde{E} < \infty$. For such an endomorphism L we define the Leray trace $Tr(L)$ by putting $Tr(L) = tr(\tilde{L})$, where the symbol tr stands for the ordinary trace. We recall that a graded vector space $\{E_q\}$ is of finite type if (i) $\dim E_q < \infty$ for all $q \in \mathbb{N}$ and (ii) $E_q = 0$ for almost all $q \in \mathbb{N}$. Now we are ready to introduce the following definition.

Definition 2.8. Let $L = \{L_q\}$ be an endomorphism of a graded vector space $E = \{E_q\}$. In addition, let $\tilde{L} = \{\tilde{L}_q\}$ be the induced endomorphism on the graded vector space $\tilde{E} = \{\tilde{E}_q\}$. We say that L is a Leray endomorphism if \tilde{E} is of finite type. For such an endomorphism L we define the generalized Lefschetz number $\Lambda(L)$ by

$$\Lambda(L) := \sum_{q=0}^{\infty} (-1)^q Tr(L_q).$$

The following proposition express one of the most important properties of the generalized Lefschetz number.

Proposition 2.3. *Let*

$$\begin{array}{ccc} E' & \xrightarrow{L} & E'' \\ L' \uparrow & \swarrow S & \uparrow L'' \\ E' & \xrightarrow{L} & E'' \end{array}$$

be a commutative diagram in the category of graded vector spaces. If one of L' , L'' is a Leray endomorphism, then so is the other; and $\Lambda(L') = \Lambda(L'')$.

Now, the Lefschetz number will be defined for admissible maps. For this purpose we need the following:

Definition 2.9. (See [11].) An admissible map $\varphi : X \rightarrow X$ is called a Lefschetz map provided for each selected pair $(p, q) \subset \varphi$ the induced homomorphism

$$q_* \circ (p_*)^{-1} : \check{H}_*(X; \mathbb{Q}) \rightarrow \check{H}_*(X; \mathbb{Q})$$

is a Leray endomorphism.

Definition 2.10. Let $\varphi : X \rightarrow X$ be a Lefschetz map. Then the Lefschetz set $\Lambda(\varphi)$ of φ is defined as follows:

$$\Lambda(\varphi) = \{\Lambda(q_* \circ (p_*)^{-1}) \mid (p, q) \subset \varphi\}.$$

The following results were established in [11].

Theorem 2.4. *Let X be an ANR and let $\varphi : X \rightarrow X$ be an admissible compact map. Then:*

- (i) φ is a Lefschetz map;
- (ii) if $\Lambda(\varphi) \neq \{0\}$, then φ has a fixed point.

Theorem 2.5. *Let $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ be two admissible maps. Then the composition $\psi \circ \varphi : X \rightarrow Z$ is an admissible map, and for every selected pair $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$ there exists a selected pair (p, q) of $\psi \circ \varphi$ such that*

$$q_{2*} \circ (p_{2*})^{-1} \circ q_{1*} \circ (p_{1*})^{-1} = q_* \circ (p_*)^{-1}. \quad (1)$$

Theorem 2.6. Let $f : X \rightarrow Y$ be a continuous selection of $\varphi : X \multimap Y$, i.e. $f(x) \in \varphi(x)$ for all $x \in X$. If $\varphi : X \multimap Y$ is an acyclic map and $(p, q) \subset \varphi$, then $q_* \circ (p_*)^{-1} = f_*$.

3. Cell-like maps

The aim of this section is to study the so-called cell-like maps. We recall some properties of cell-like maps that will be needed in the next sections. First, following [6] we recall the concept of movability.

Definition 3.1. Let X be an ANR and let $X_0 \subset X$ be a closed subset. We say that X_0 is movable in X provided every neighborhood U of X_0 admits a neighborhood U' of X_0 , $U' \subset U$, such that for every neighborhood U'' of X_0 , $U'' \subset U$, there exists a homotopy $H : U' \times [0, 1] \rightarrow U$ with $H(x, 0) = x$ and $H(x, 1) \in U''$, for any $x \in U'$ (in other words, sufficiently small neighborhoods of X_0 can be deformed arbitrarily close to X_0).

Definition 3.2. Let X be a compact metric space. We say that X is movable provided there exists $Z \in \text{ANR}$ and an embedding $e : X \rightarrow Z$ such that $e(X)$ is movable in Z .

Let us notice that the property of being movable is an absolute property, that is if A is a movable set in some ANR X and $j : A \hookrightarrow X'$ is an embedding into an ANR X' , then $j(A)$ is movable in X' (see [5] or [3]). We shall make use of the following result from [3].

Lemma 3.1. Let X be an ANR and let $X_0 \subset X$ be a compact absolute approximative neighborhood retract in the sense of Clapp. Then X_0 is movable in X .

The following two concepts will play a crucial role in our considerations.

Definition 3.3. A compact metric space K is called cell-like if there is $Z \in \text{ANR}$ and an embedding $e : K \rightarrow Z$ such that $e(K)$ is contractible in each of its neighborhoods in Z .

Clearly any compact contractible or convex set is cell-like. It is easy to see that cell-likeness is an absolute property, i.e. given a cell-like space K , an arbitrary ANR Z and an embedding $e : K \rightarrow Z$, $e(K)$ is contractible in each of its neighborhoods in Z . The Cartesian product of two cell-like sets is cell-like. The famous result due to D.M. Hyman (see [18]) states that if a compact space K is cell-like, then there exists a decreasing sequence of contractible compacta $\{K_n\}$ containing K as a closed subspace such that $K = \bigcap_{n=1}^{\infty} K_n$. This fact implies that cell-like sets are acyclic with respect to any continuous homology theory (e.g. the Čech homology).

Definition 3.4. A proper map $f : X \rightarrow Y$ is called a cell-like map (f is a CE-map) provided that $f^{-1}(y)$ is a cell-like set for every $y \in Y$.

Let us notice that any CE-map is a Vietoris map, but the converse is not true, i.e. a Vietoris map need not be a CE-map. Now we shall collect some facts concerning the above notions.

Theorem 3.1. (See [4] or [26].) If $X \in \text{ANR}$ and Y is a finite-dimensional metric space and if a map $f : X \rightarrow Y$ satisfies the following condition $f^{-1}(y) \in \text{AR}$ for every point $y \in Y$, then $Y \in \text{ANR}$.

The above theorem does not remain true if we omit the hypothesis that the covering dimension of the space Y is finite. Some examples will be presented below.

The following theorem was established by W.E. Haver (see [17]).

Theorem 3.2. A cell-like map $f : X \rightarrow Y$ between compact ANRs is a homotopy equivalence.

The restrictive assumptions on X and Y cannot be omitted. This was first shown by J.L. Taylor (see [27]). He used a metric continuum X defined by J.F. Adams (see [1]). The space X is constructed as follows.

First, Adams has defined a compact polyhedron Y (of the form $S^k \sqcup_g B^{k+1}$), an integer $r \geq 1$ and a continuous map $h_0 : S^r Y \rightarrow Y$ from the r -fold suspension of Y into Y such that for every n the following composition

$$h_0 \circ h_1 \circ \dots \circ h_{n-1} \circ h_n : S^{(n+1)r} Y \rightarrow Y$$

is not homotopic to a constant map, where $h_{i+1} : S^{(i+2)r} Y \rightarrow S^{(i+1)r} Y$ is the r -fold suspension $S^r h_i : S^r(S^{(i+1)r} Y) \rightarrow S^r(S^{ir} Y)$ of $h_i : S^{(i+1)r} Y \rightarrow S^{ir} Y$, for $0 \leq i \leq n-1$. Next considering the inverse sequence

$$Y \xleftarrow{h_0} S^r Y \xleftarrow{h_1} S^{2r} Y \xleftarrow{h_2} \dots \xleftarrow{h_m} S^{(m+1)r} Y \xleftarrow{\dots}$$

he defined a space X to be the inverse limit of the above inverse sequence.

Having such a space X , Taylor constructed a cell-like map $f: X \rightarrow I^\infty$ onto the Hilbert cube I^∞ (the Cartesian product of countable infinity of copies of the interval $[0, 1]$) which is not a shape equivalence. Next, using the Taylor example, J.E. Keesling (see [19]) constructed a cell-like map $h: I^\infty \rightarrow Y$ of the Hilbert cube onto a non-movable metric continuum Y (in particular, Y is not an ANR).

It should be remarked that R.J. Daverman and J.J. Walsh (see [8]) used the Taylor example (see [27]) to produce another cell-like maps with interesting properties.

In addition, J. van Mill observed that there exists a cell-like map $H: (I^\infty)^\infty \rightarrow Y^\infty$ such that no non-empty open subset of Y^∞ is contractible in Y^∞ (see [29]). For this purpose, it is enough to construct a cell-like map $H: (I^\infty)^\infty \rightarrow Y^\infty$ by $H = h \times h \times h \times \dots$, where for any space X the symbol X^∞ stands for the countable infinite product of copies X (let us observe that a space Y^∞ is not movable since a space Y is not movable) and h is as above.

Moreover, by using the Keesling example, J. van Mill has produced a cell-like map $g: I^\infty \rightarrow Z$ such that Z is not movable (and therefore a space Z is not an ANR) and each fiber $g^{-1}(z)$, $z \in Z$, is an AR (see [28]).

Using the above constructions, one can provide further examples of cell-like maps whose the domain spaces are non-contractible ANRs and whose range spaces are not ANRs. Now we are going to present two such examples.

Example 3.1. Let Y be a noncontractible compact ANR and let $id_Y: Y \rightarrow Y$ be the identity map. Assume further that $F: X \rightarrow Z$ is a cell-like map such that X is a contractible compact ANR and Z is not an ANR. Then a map $id_Y \times F: Y \times X \rightarrow Y \times Z$ defined by $(id_Y \times F)(y, x) = \{y\} \times \{F(x)\}$ is a cell-like map with the required properties.²

The second example is much more interesting and it is a slight modification of the example given by J.E. Keesling in [19].

Example 3.2. Let $f: X \rightarrow I^\infty$ be a cell-like map constructed by J.L. Taylor. Let $I_0^\infty := I^\infty$. Since X is compact, we can consider X as a subset of I_0^∞ with $I_0^\infty \setminus X \neq \emptyset$ (recall that X is not contractible). Let

$$\pi: I_0^\infty \sqcup I^\infty \rightarrow I_0^\infty \sqcup_f I^\infty$$

be the natural quotient mapping. Let $i: I_0^\infty \hookrightarrow I_0^\infty \sqcup_f I^\infty$ be the inclusion. Then a function $F: I_0^\infty \rightarrow I_0^\infty \sqcup_f I^\infty$ defined by $F := \pi \circ i$ is a cell-like map. From Proposition 2.2 it follows that

$$F(I_0^\infty \setminus X): I_0^\infty \setminus X \xrightarrow{\approx} F(I_0^\infty \setminus X) \quad (2)$$

is a homeomorphism and that $F(I_0^\infty \setminus X)$ is open in $I_0^\infty \sqcup_f I^\infty$. Furthermore, it is easy to see that

$$F(I_0^\infty \setminus X) \cap F(X) = \emptyset. \quad (3)$$

Let us take a point $x = \{x_i\} \in I_0^\infty \setminus X$. Since $I_0^\infty \setminus X$ is open, there exists an $r > 0$ such that $B(x, r) \subset I_0^\infty \setminus X$.³ Consequently, there exist $\varepsilon > 0$, a positive integer $k > 1$ and open sets $B(x_i, \varepsilon) \subset [0, 1]$, where $i = 1, \dots, k$, such that

$$\left(\prod_{i=1}^k B(x_i, \varepsilon) \right) \times \left(\prod_{i=k+1}^\infty [0, 1] \right) \subset B(x, r).$$

Now, let $[a_i, b_i]$ be a closed interval contained in $B(x_i, \varepsilon)$, for $i = 1, 2, \dots, k$ (we can assume that $0 < a_i < b_i < 1$, for $i = 1, 2, \dots, k$). Then,

$$\left(\prod_{i=1}^k [a_i, b_i] \right) \times \left(\prod_{i=k+1}^\infty [0, 1] \right) \subset B(x, r).$$

Let us put

$$B := \left(\prod_{i=1}^k (a_i, b_i) \right) \times \left(\prod_{i=k+1}^\infty [0, 1] \right) \subset I_0^\infty \setminus X. \quad (4)$$

Let us observe that $I_0^\infty \setminus B$ is an ANR. Indeed, this follows from the following equality

$$I_0^\infty \setminus B = \left([0, 1]^k \setminus \left(\prod_{i=1}^k (a_i, b_i) \right) \right) \times \left(\prod_{i=k+1}^\infty [0, 1] \right)$$

² The space $Y \times Z$ is not an ANR since the following result is true: the Cartesian product $X = X_1 \times X_2$ is an ANR if and only if every factor X_i is an ANR, for $i = 1, 2$ —see [4].

³ Recall that a metric d on the Hilbert cube I^∞ is defined by $d(\{x_i\}, \{y_i\}) = \sum_{i=1}^\infty \frac{1}{2^i} |x_i - y_i|$ for all $\{x_i\}, \{y_i\} \in I^\infty$.

and the fact that $([0, 1]^k \setminus (\prod_{i=1}^k (a_i, b_i)))$ is an ANR, where $[0, 1]^k$ stands for the Cartesian product of k copies of the interval $[0, 1]$. Moreover, it is easy to see that the space $I_0^\infty \setminus B$ has the homotopy type of a $(k-1)$ -sphere \mathbb{S}^{k-1} in \mathbb{R}^k , and hence it is a noncontractible space. Taking into account (2)–(4), one obtains

$$\begin{aligned} F(I_0^\infty \setminus B) &= F((I_0^\infty \setminus X) \cup X) \\ &= F((I_0^\infty \setminus X) \setminus B) \cup F(X) \\ &= (F(I_0^\infty \setminus X) \setminus F(B)) \cup F(X) \\ &= (F(I_0^\infty \setminus X) \setminus F(B)) \cup (F(X) \setminus F(B)) \\ &= (F(I_0^\infty \setminus X) \cup F(X)) \setminus F(B) \\ &= F(I_0^\infty) \setminus F(B) = (I_0^\infty \sqcup_f I^\infty) \setminus F(B). \end{aligned}$$

Consequently, we can define a function

$$\tilde{F}: I_0^\infty \setminus B \rightarrow (I_0^\infty \sqcup_f I^\infty) \setminus F(B) \quad (5)$$

by $\tilde{F}(x) = F(x)$ for all $x \in I_0^\infty \setminus B$. Since F is a cell-like map, we deduce that \tilde{F} is also a cell-like map. Now let us observe that $(I_0^\infty \sqcup_f I^\infty) \setminus F(B)$ is not an ANR. Indeed, if $(I_0^\infty \sqcup_f I^\infty) \setminus F(B)$ were an ANR, then

$$I_0^\infty \sqcup_f I^\infty = ((I_0^\infty \sqcup_f I^\infty) \setminus F(B)) \cup F(\bar{B})$$

would also be an ANR by Proposition 2.1, because $F(\bar{B})$ and $F(\partial B)$ are ANRs and the following condition holds

$$((I_0^\infty \sqcup_f I^\infty) \setminus F(B)) \cap F(\bar{B}) = ((I_0^\infty \sqcup_f I^\infty) \setminus F(B)) \cap \overline{F(B)} = \partial F(B) = F(\partial B).$$

However, a space $I_0^\infty \sqcup_f I^\infty$ is not an ANR since $I_0^\infty \sqcup_f I^\infty$ is not movable (the proof of non-movability of $I_0^\infty \sqcup_f I^\infty$ is contained in the proof of Theorem 4 in [19]). Thus, we have proved that $(I_0^\infty \sqcup_f I^\infty) \setminus F(B)$ is not an ANR, as required.

Remark 3.1. For more information on cell-like maps and related results, we refer the reader to [21,29–31].

4. Absolute neighborhood multi-retracts

The main goal of this section is to generalize the notions of absolute retract and absolute neighborhood retract. In the next section we shall use these notions in order to obtain new fixed point results of Lefschetz type for admissible maps.

We begin by introducing the concept of an *mr*-map.

Definition 4.1. A map $r: X \rightarrow Y$ of a space X onto a space Y is said to be an *mr*-map if there is an admissible map $\varphi: Y \multimap X$ such that $r \circ \varphi = id_Y$.

An admissible map φ satisfying the following condition $r \circ \varphi = id_Y$ will be called an admissible right inverse of r . Now we establish some properties of *mr*-maps in the following lemmas.

Lemma 4.1. The composition of two *mr*-maps is an *mr*-map.

Proof. Let $f_1: X \rightarrow Y$ and $f_2: Y \rightarrow Z$ be two *mr*-maps and let $\varphi_1: Y \multimap X$ and $\varphi_2: Z \multimap Y$ be their admissible right inverses. Let

$$f := f_2 \circ f_1: X \rightarrow Z \quad \text{and} \quad \varphi := \varphi_1 \circ \varphi_2: Z \multimap X.$$

Then, by Theorem 2.2, φ is an admissible map. In addition,

$$f \circ \varphi = f_2 \circ f_1 \circ \varphi_1 \circ \varphi_2 = f_2 \circ id_Y \circ \varphi_2 = f_2 \circ \varphi_2 = id_Z.$$

Hence the multivalued map φ is an admissible right inverse of f , which completes the proof. \square

Lemma 4.2. If $r: X \rightarrow Y$ is an *mr*-map, $Y_0 \subset Y$ and $X_0 = r^{-1}(Y_0)$, then the restriction $r_0: X_0 \rightarrow Y_0$ of r is an *mr*-map.

Proof. Let $\varphi: Y \multimap X$ be an admissible right inverse of r . Since $r \circ \varphi = id_Y$ and $X_0 = r^{-1}(Y_0)$, it follows that $\varphi^{-1}(X_0) = Y_0$. Consequently, by Lemma 2.1, the restriction $\varphi_0: Y_0 \multimap X_0$ of φ is an admissible map. It is clear that $r_0 \circ \varphi_0 = id_{Y_0}$, which completes the proof. \square

Now we are ready to introduce the following definitions.

Definition 4.2. A metric space X is called an absolute multi-retract (notation: $X \in AMR$) provided there exists a normed linear space E and an mr -map $r: E \rightarrow X$ from E onto X .

Definition 4.3. A metric space X is called an absolute neighborhood multi-retract (notation: $X \in ANMR$) provided there exists an open subset U of some normed linear space E and an mr -map $r: U \rightarrow X$ from U onto X .

In order to illustrate Definitions 4.2 and 4.3, let us consider the following examples.

Example 4.1. Let $f: I^\infty \rightarrow Y$ be any cell-like map such that Y is not an ANR (some examples of such maps are provided in the previous section). We will show that Y is an AMR . Since I^∞ is an AR , there exists a normed linear space E and an r -map $\tilde{r}: E \rightarrow I^\infty$. Hence, there exists a continuous map $\tilde{s}: I^\infty \rightarrow E$ such that $\tilde{r} \circ \tilde{s} = id_{I^\infty}$. Now we define $r: E \rightarrow Y$ and $\varphi: Y \rightarrow E$ as follows:

$$\begin{aligned} r(x) &:= f(\tilde{r}(x)) \quad \text{for all } x \in E, \\ \varphi(y) &:= \tilde{s}(f^{-1}(y)) \quad \text{for all } y \in Y. \end{aligned}$$

Since φ is admissible (see Definition 2.2) and $r(\varphi(y)) = f(\tilde{r}(\tilde{s}(f^{-1}(y)))) = f(f^{-1}(y)) = y$ for all $y \in Y$, we conclude that $r: E \rightarrow Y$ is an mr -map, which completes the proof that Y is an AMR .

Example 4.2. Let $f: I^\infty \rightarrow Y$ be as in Example 4.1 and let \mathbb{S}^1 be the unit circle in \mathbb{R}^2 . We will show that

- (a) $\mathbb{S}^1 \times Y \in ANMR$,
- (b) $\mathbb{S}^1 \times Y \notin AMR$,
- (c) $\mathbb{S}^1 \times Y \notin ANR$.

Let $r: E \rightarrow Y$ and $\varphi: Y \rightarrow E$ be as in Example 4.1. Furthermore, there exists an open subset U of \mathbb{R}^2 and an r -map $r': U \rightarrow \mathbb{S}^1$. Since r' is an r -map, there exists a continuous map $s': \mathbb{S}^1 \rightarrow U$ such that $r' \circ s' = id_{\mathbb{S}^1}$. Consider two product maps

$$r' \times r: U \times E \rightarrow \mathbb{S}^1 \times Y \quad \text{and} \quad s' \times \varphi: \mathbb{S}^1 \times Y \rightarrow U \times E.$$

Observe that Lemma 2.2 implies that the product map $s' \times \varphi: \mathbb{S}^1 \times Y \rightarrow U \times E$ is admissible. Moreover, one has

$$(r' \times r) \circ (s' \times \varphi) = (r' \circ s') \times (r \circ \varphi) = id_{\mathbb{S}^1} \times id_Y = id_{\mathbb{S}^1 \times Y},$$

which implies that $\mathbb{S}^1 \times Y \in ANMR$. Now we shall prove that $\mathbb{S}^1 \times Y \notin AMR$. For this purpose, let us observe that from Theorem 2.1 it follows that Y is acyclic (notice that I^∞ is contractible!). Consequently, the Künneth theorem⁴ for the Čech homology functor implies that

$$\check{H}_1(\mathbb{S}^1 \times Y; \mathbb{Q}) = (\check{H}_0(\mathbb{S}^1; \mathbb{Q}) \otimes \check{H}_1(Y; \mathbb{Q})) \oplus (\check{H}_1(\mathbb{S}^1; \mathbb{Q}) \otimes \check{H}_0(Y; \mathbb{Q})) = \mathbb{Q} \otimes \mathbb{Q} = \mathbb{Q}. \quad (6)$$

Consequently, in view of (6) and Proposition 5.1 (see below), one can easily conclude that $\mathbb{S}^1 \times Y \notin AMR$. What is left is to show that $\mathbb{S}^1 \times Y \notin ANR$. For this purpose, let us recall that $Y \notin ANR$. Consequently, taking into account the fact that a space $\mathbb{S}^1 \times X$ is an ANR if and only if X is an ANR (see [4]), we deduce that $\mathbb{S}^1 \times Y$ is not an ANR , as required.

Example 4.3. Let $\tilde{F}: I_0^\infty \setminus B \rightarrow (I_0^\infty \sqcup_f I^\infty) \setminus F(B)$ be as in Example 3.2. It was shown in Example 3.2 that $(I_0^\infty \sqcup_f I^\infty) \setminus F(B) \notin ANR$. By using arguments similar to those used in Examples 4.1 and 4.2, we can show that

- (a) $(I_0^\infty \sqcup_f I^\infty) \setminus F(B) \in ANMR$,
- (b) $(I_0^\infty \sqcup_f I^\infty) \setminus F(B) \notin AMR$.

The following diagram illustrates the relationships between the introduced concepts:

$$\begin{array}{ccc} AR & \subset & ANR \\ \cap & & \cap \\ AMR & \subset & ANMR. \end{array}$$

Taking into account Examples 4.1–4.3, we infer that all the above inclusions are proper (it is clear that the inclusion $AR \subset ANR$ is proper).

Below we collect some simple but important properties of $AMRs$ and of $ANMRs$. The following proposition allows us to provide a large variety of examples of $AMRs$ and of $ANMRs$.

⁴ The Künneth theorem asserts that for every two compact spaces of finite type X_1 and X_2 there is a linear isomorphism $L: \check{H}_*(X_1 \times X_2; \mathbb{Q}) \xrightarrow{\sim} \check{H}_*(X_1; \mathbb{Q}) \otimes \check{H}_*(X_2; \mathbb{Q})$ (see e.g. [13]). Recall that a space Z is of finite type provided the graded vector space $\check{H}_*(Z; \mathbb{Q})$ is of finite type.

Proposition 4.1. *Let $X \in \text{ANMR}$ ($X \in \text{AMR}$) and let $p : X \rightarrow Y$ be a Vietoris map. Then $Y \in \text{ANMR}$ ($Y \in \text{AMR}$).*

Proof. Assume that $X \in \text{ANMR}$. Let $p : X \rightarrow Y$ be a Vietoris map. Since $X \in \text{ANMR}$, there exists an open subset U of some normed linear space E and an mr -map $r : U \rightarrow X$ from U onto X . Let $s : X \rightarrow U$ be an admissible right inverse of r . Now let us define $r' : U \rightarrow Y$ and $s' : Y \rightarrow U$ by $r' = p \circ r$ and $s' = s \circ p^{-1}$, respectively. From Theorem 2.2 it follows that s' is an admissible map. To complete the proof, it suffices to observe that

$$r' \circ s' = (p \circ r) \circ (s \circ p^{-1}) = p \circ (r \circ s) \circ p^{-1} = p \circ p^{-1} = \text{id}_Y.$$

Consequently, we have proved that $Y \in \text{ANMR}$. The proof for the case $X \in \text{AMR}$ is entirely analogous and therefore we leave it to the reader. \square

Corollary 4.1. *Let $X \in \text{ANR}$ ($X \in \text{AR}$) and let $p : X \rightarrow Y$ be a Vietoris map. Then $Y \in \text{ANMR}$ ($Y \in \text{AMR}$).*

In particular, we obtain the following corollary.

Corollary 4.2. *Let $X \in \text{ANR}$ ($X \in \text{AR}$) and let $p : X \rightarrow Y$ be a CE -map. Then $Y \in \text{ANMR}$ ($Y \in \text{AMR}$).*

Remark 4.1. It should be noted that a space Y from Corollaries 4.1 and 4.2 need not be an ANR (moreover, a space Y need not be an AANR_C —see Lemma 3.1). This follows from the examples provided in the section devoted to cell-like maps. So, to sum up, the examples given in the previous section and Corollary 4.2 illustrate the fact that the class of ANMRs (AMRs) is significantly larger than that of the class of ANRs (ARs).

Now we are going to show that AMRs admit the following characterization.

Proposition 4.2. *A space X is an AMR if and only if there exists a metric space Z^5 and a Vietoris map $p : Z \rightarrow X$ which factors through a normed linear space E , i.e. there are two continuous maps α and β such that the following diagram*

$$\begin{array}{ccc} & Z & \\ \alpha \swarrow & & \downarrow p \\ E & \xrightarrow{\beta} & X \end{array}$$

is commutative.

Proof. Assume that $X \in \text{AMR}$. Then there exists a normed linear space E , a map $r : E \rightarrow X$ and an admissible map $\varphi : X \rightarrow E$ such that

$$r(\varphi(x)) = x, \quad (7)$$

for any $x \in X$. Since φ is admissible, it follows that there exists a selected pair $X \xleftarrow{\tilde{p}} Z \xrightarrow{\tilde{q}} E$ such that

$$\varphi(x) = \tilde{q}(\tilde{p}^{-1}(x)), \quad (8)$$

for all $x \in X$. Consequently, taking into account (7) and (8), we obtain the following commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{q}} & E \\ \tilde{p} \downarrow & & \downarrow r \\ X & \xrightarrow{\text{id}_X} & X. \end{array}$$

Thus, we can define α , β and p as follows: $\alpha := \tilde{q}$, $\beta := r$ and $p := \tilde{p}$. It is clear that $p = \beta \circ \alpha$, as desired.

Now we shall prove the opposite implication. For this purpose, assume that there exists a normed linear space E and two maps $\alpha : Z \rightarrow E$, $\beta : E \rightarrow X$ such that $\beta \circ \alpha : Z \rightarrow X$ is a Vietoris map. Then define $r : E \rightarrow X$, $p : Z \rightarrow X$ and $q : Z \rightarrow E$ as follows: $r := \beta$, $p := \beta \circ \alpha$ and $q := \alpha$. Let $\varphi : X \rightarrow E$ be given by the formula $\varphi(x) := q(p^{-1}(x))$, for all $x \in X$. Then it is not hard to see that φ is an admissible right inverse of r , which completes the proof. \square

We also have a similar characterization of ANMRs .

⁵ Let us notice that the space Z is not assumed to be an ANR .

Proposition 4.3. A space X is an ANMR if and only if there exists a metric space Z and a Vietoris map $p : Z \rightarrow X$ which factors through an open subset U of some normed linear space E , i.e. there are two continuous maps α and β making the diagram

$$\begin{array}{ccc} & Z & \\ \alpha \swarrow & & \searrow p \\ U & \xrightarrow{\beta} & X \end{array}$$

commutative.

Proof. The proof of this proposition is completely parallel to the proof of Proposition 4.2, and therefore we leave it to the reader. \square

Example 5.2, which will be given in the last section of this paper, will show that if a function $p : X \rightarrow Y$ defined on an ANR is not a Vietoris map, then the image of the space X under p need not be an ANMR. Further, the following example shows that there exists an mr -map $r : U \rightarrow X$ which is not an r -map.

Example 4.4. Let $\tilde{F} : \tilde{X} \rightarrow \tilde{Y}$ be, for instance, as in Example 3.2 (see (5)). Since \tilde{X} is an ANR, there exists an open subset U of some normed linear space E and an r -map $r' : U \rightarrow \tilde{X}$ from U onto \tilde{X} . Then, there exists a function $s' : \tilde{X} \rightarrow U$ such that $r' \circ s' = id_{\tilde{X}}$. Let us define a map $r : U \rightarrow \tilde{Y}$ by $r := \tilde{F} \circ r'$. Let us observe that the map r is an mr -map. Indeed, let $\varphi : \tilde{Y} \rightarrow U$ be defined by $\varphi(y) := s'(\tilde{F}^{-1}(y))$ for all $y \in \tilde{Y}$. It is clear that φ is an admissible map and that $r \circ \varphi = id_{\tilde{Y}}$. So, we conclude that r is an mr -map. However, $r : U \rightarrow \tilde{Y}$ is not an r -map. This follows directly from Definition 2.4 and the fact that \tilde{Y} is not an ANR.

We complete this section with some simple but important propositions on ANMRs.

Proposition 4.4. Let $X \in ANMR$ and let U be an open subset of X . Then $U \in ANMR$.

Proof. It follows from Lemma 4.2. \square

Proposition 4.5. If $X_1, X_2 \in ANMR$ ($X_1, X_2 \in AMR$), then $(X_1 \times X_2) \in ANMR$ ($X_1 \times X_2 \in AMR$).

Proof. This statement follows directly from Definitions 4.2, 4.3 and Lemma 2.2. \square

5. Fixed point results

In this section we shall prove some fixed point results of Lefschetz type for admissible maps defined on ANMRs. We first show that any compact AMR is acyclic and that any compact ANMR is of finite type. For this purpose, we need to recall the following definition and lemma (see [11]).

Definition 5.1. Two admissible maps $\varphi, \psi : X \rightarrow Y$ are called homotopic (written $\varphi \sim \psi$) provided there exists an admissible map $\chi : X \times [0, 1] \rightarrow Y$ such that

$$\chi(x, 0) = \varphi(x) \quad \text{and} \quad \chi(x, 1) = \psi(x) \quad \text{for every } x \in X.$$

Lemma 5.1. Let $\varphi, \psi : X \rightarrow Y$ be two admissible maps. Then $\varphi \sim \psi$ implies that there exist selected pairs $(p, q) \subset \varphi$ and $(\tilde{p}, \tilde{q}) \subset \psi$ such that

$$q_* \circ (p_*)^{-1} = \tilde{q}_* \circ (\tilde{p}_*)^{-1}.$$

Proposition 5.1. If $X \in AMR$, then X is acyclic with respect to the Čech homology functor with compact carriers and rational coefficients.

Proof. Since $X \in AMR$, there exists an mr -map $r : E \rightarrow X$ and an admissible map $\varphi : X \rightarrow E$ such that $r \circ \varphi = id_X$, where E is a normed linear space. Let $(p, q) \subset \varphi$ be a selected pair of φ . Then $(r \circ q, p) \subset id_X$ is a selected pair of id_X . Consequently, Theorem 2.6 implies that $(r \circ q)_* \circ (p_*)^{-1} = id_{\check{H}_*(X; \mathbb{Q})}$. Moreover, one has $(r \circ q)_* = r_* \circ q_*$. Thus, $q_* \circ (p_*)^{-1} : \check{H}_*(X; \mathbb{Q}) \rightarrow \check{H}_*(E; \mathbb{Q})$ is a monomorphism. From this it follows that a space X is acyclic, which completes the proof. \square

Proposition 5.2. Let X be a compact metric space. If $X \in ANMR$, then X is of finite type.

Proof. Let $X \in \text{ANMR}$. Then there exists a continuous map $r: U \rightarrow X$ and an admissible map $\varphi: X \multimap U$ such that $r \circ \varphi = \text{id}_X$, where U is an open subset of a normed linear space E . Let us observe that $\varphi(X) \subset U$ is compact. Let $i: \varphi(X) \hookrightarrow U$ be the inclusion. Then, it follows from Theorem 2.3 that there exists a compact polyhedron $K' \subset U$ and a continuous map $i': \varphi(X) \rightarrow U$ such that the following conditions are satisfied:

- (i) $i'(\varphi(X)) \subset K'$,
- (ii) the maps $i, i': \varphi(X) \rightarrow U$ are homotopic.

Let $\varphi': X \multimap U$ be defined by $\varphi' := i' \circ \varphi$. From Theorem 2.2 it follows that φ' is an admissible map. Furthermore, φ' is easily seen to be homotopic to $\varphi = i \circ \varphi$ (in the sense of Definition 5.1). Consequently, Lemma 5.1 implies that there exist selected pairs $(p, q) \subset \varphi$ and $(p', q') \subset \varphi'$ such that

$$q_* \circ (p_*)^{-1} = q'_* \circ (p'_*)^{-1}.$$

Since $r \circ \varphi = \text{id}_X$, we infer that $(r \circ q, p) \subset \text{id}_X$. Thus, Theorem 2.6 implies that $(r \circ q)_* \circ (p_*)^{-1} = \text{id}_{\check{H}_*(X; \mathbb{Q})}$, and since $(r \circ q)_* = r_* \circ q_*$, we conclude that $q_* \circ (p_*)^{-1}: \check{H}_*(X; \mathbb{Q}) \rightarrow \check{H}_*(U; \mathbb{Q})$ is a monomorphism. Hence $q'_* \circ (p'_*)^{-1}: \check{H}_*(X; \mathbb{Q}) \rightarrow \check{H}_*(U; \mathbb{Q})$ is a monomorphism.

On the other hand, since $\varphi'(X) \subset K'$, where K' is of finite type, it follows that q' factors through the space K' . More precisely, the following diagram is commutative

$$\begin{array}{ccc} X & \xleftarrow{p'} \Gamma' & \xrightarrow{q'} U \\ & \searrow \tilde{q} & \uparrow i \\ & & K' \end{array}$$

where $i: K' \hookrightarrow U$ is the inclusion and \tilde{q} is given by $\tilde{q}(z) = q'(z)$, for all $z \in \Gamma'$. Thus, the above considerations imply that X is of finite type. This completes the proof. \square

The following example of M.H. Clapp in [7] is an AANR_C that is not an ANMR .

Example 5.1. Let

$$A_n := \{(x, y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = 1/n^2\},$$

for any integer $n \geq 1$. Let $X := \bigcup_{n=1}^{\infty} A_n$. It was shown in [7] that X is an AANR_C . On the other hand, since X is not of finite type, we infer from Proposition 5.2 that X is not an ANMR .

Consequently, we deduce from Remark 4.1 and Example 5.1 that there is no connection between the class of approximative absolute neighborhood retracts and the class of absolute neighborhood multi-retracts.

We shall state now a generalization of the Lefschetz fixed point theorem to ANMRs .

Theorem 5.1. Let $X \in \text{ANMR}$ and let $\psi: X \multimap X$ be a compact admissible map. Then:

- (i) ψ is a Lefschetz map;
- (ii) if $\Lambda(\psi) \neq \{0\}$, then $\text{Fix}(\psi) \neq \emptyset$.

Proof. The proof is based upon ideas found in [11,15]. Since $X \in \text{ANMR}$, there exists a continuous map $r: U \rightarrow X$ and an admissible map $\varphi: X \multimap U$ such that $r \circ \varphi = \text{id}_X$, where U is an open subset of a normed linear space E . Then, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & U \\ \psi \downarrow & \searrow \psi \circ r & \downarrow \varphi \circ \psi \circ r \\ X & \xrightarrow{\varphi} & U \end{array}$$

By Theorem 2.2, the multivalued map $\varphi \circ \psi \circ r$ is admissible and compact. Let (p, q) be a selected pair of ψ . Let (p_1, q_1) be a selected pair of φ . Now we shall show that $q_* \circ (p_*)^{-1}$ is a Leray endomorphism. From Theorem 2.6 we infer that for every selected pair $(p_2, q_2) \subset r$ we have $q_{2*} \circ (p_{2*})^{-1} = r_*$. Using Theorem 2.5, we conclude that there exists a selected pair $(\bar{p}, \bar{q}) \subset \varphi \circ \psi \circ r$ such that

$$\bar{q}_* \circ (\bar{p}_*)^{-1} = q_{1*} \circ (p_{1*})^{-1} \circ q_* \circ (p_*)^{-1} \circ q_{2*} \circ (p_{2*})^{-1}.$$

Now observe that Theorem 2.4 implies that $\bar{q}_* \circ (\bar{p}_*)^{-1}$ is a Leray endomorphism. In addition, Theorems 2.5 and 2.6 imply that

$$q_{2*} \circ (p_{2*})^{-1} \circ q_{1*} \circ (p_{1*})^{-1} = id_{\check{H}_*(X; \mathbb{Q})}.$$

Hence, it is not hard to see that the following diagram

$$\begin{array}{ccc} \check{H}_*(X; \mathbb{Q}) & \xrightarrow{q_{1*}(p_{1*})^{-1}} & \check{H}_*(U; \mathbb{Q}) \\ \uparrow q_*(p_*)^{-1} & \swarrow q_*(p_*)^{-1}r_* & \uparrow \bar{q}_*(\bar{p}_*)^{-1} \\ \check{H}_*(X; \mathbb{Q}) & \xrightarrow{q_{1*}(p_{1*})^{-1}} & \check{H}_*(U; \mathbb{Q}) \end{array}$$

is commutative. Consequently, taking into account Proposition 2.3 and Theorem 2.4, we deduce that $q_* \circ (p_*)^{-1}$ is a Leray endomorphism. Thus, we have proved that ψ is a Lefschetz map. In order to prove (ii), assume that $\Lambda(\psi) \neq \{0\}$. Then, there exists a selected pair $(p, q) \subset \psi$ such that $\Lambda(q_* \circ (p_*)^{-1}) \neq 0$. Now, by the same reasoning as before, we conclude that there exists a selected pair $(\bar{p}, \bar{q}) \subset \varphi \circ \psi \circ r$ such that $\Lambda(\bar{q}_* \circ (\bar{p}_*)^{-1}) = \Lambda(q_* \circ (p_*)^{-1}) \neq 0$. Consequently, from Theorem 2.4 it follows that there exists a point $x_0 \in U$ such that $x_0 \in \varphi(\psi(r(x_0)))$. Since $r(\varphi(x)) = x$ for any $x \in X$, we infer that $r(x_0) \in \psi(r(x_0))$, which completes the proof. \square

As a direct consequence of Theorem 5.1, we obtain the following:

Theorem 5.2. *Let X be an acyclic ANMR and let $\psi : X \multimap X$ be a compact admissible map. Then $\Lambda(\psi) = \{1\}$ and hence $\text{Fix}(\psi) \neq \emptyset$.*

In particular, we obtain the following corollary.

Corollary 5.1. *If $X \in \text{AMR}$ and $\psi : X \multimap X$ is a compact admissible map, then ψ has a fixed point.*

Now, using the Knill example (see [20]), we will show that if the hypothesis in Proposition 4.1 that $p : X \rightarrow Y$ is a Vietoris map is not satisfied, then the assertion of Proposition 4.1 need not be true.

Example 5.2. Let $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ be the unit circle. Let $\alpha : [2, \infty) \rightarrow \mathbb{R}^2$ be a function defined as follows:

$$\alpha(t) = ((1 + 2^{-t}) \cos(t), (1 + 2^{-t}) \sin(t))$$

for any $t \geq 2$. Let $X := \alpha([2, \infty))$ and let $Y := (\mathbb{S}^1 \cup X) \subset \mathbb{R}^2$. Let $CY = (\mathbb{S}^1 \cup X) \times [0, 1] / \sim$ be the cone over Y with the vertex $[y, 1]$, where $[y, 1]$ denotes the equivalence class of $(y, 1)$. It is not hard to see that CY is contractible to the point $[y, 1]$.⁶ Now, let us define a function

$$p : (\mathbb{S}^1 \cup ([2, \infty) \times \{0\})) \times [0, 1] \rightarrow CY$$

by $p(z, s) = [\beta(z), s]$, where $\beta : \mathbb{S}^1 \cup ([2, \infty) \times \{0\}) \rightarrow \mathbb{S}^1 \cup X$ is given by the following formula

$$\beta(z) = \begin{cases} z & \text{if } z \in \mathbb{S}^1, \\ \alpha(t) & \text{if } z = (t, 0) \in [2, \infty) \times \{0\}. \end{cases}$$

Observe that a set $p^{-1}([y, s])$ is acyclic for any $[y, s] \in CY$ except for the point $[y, 1] \in CY$. So, this observation implies that the function p is not a Vietoris map. In addition, it was shown by R.J. Knill [20] that CY is not a Lefschetz space.⁷ Consequently, by using this fact and Theorem 5.2, we conclude that CY is not an ANMR, as required.

Now we want to extend the Lefschetz fixed point theorem onto a class of noncompact maps. For this purpose we need to introduce an additional notion.

Definition 5.2. (See [10] or [13].) A multivalued map $\varphi : X \multimap X$ is called a compact absorbing contraction if there exists an open subset $U \subset X$ such that the following two conditions are satisfied:

- (i) $\varphi(U) \subset U$ and the map $\tilde{\varphi} : U \multimap U$, $\tilde{\varphi}(x) = \varphi(x)$ for every $x \in U$, is compact;
- (ii) for every $x \in X$, there exists a natural number n_x such that $\varphi^{n_x}(x) \subset U$.

⁶ For this purpose, it is enough to define a homotopy $H : CY \times [0, 1] \rightarrow CY$ as follows $H([y, s], t) = [y, (1-t)s + t]$, for all $[y, s] \in CY$ and $t \in [0, 1]$.

⁷ Recall that a space X is said to be a Lefschetz space if any compact map $f : X \rightarrow X$ satisfies the following conditions: (a) $\Lambda(f)$ is well defined and (b) $\Lambda(f) \neq 0$ implies that f has a fixed point.

In what follows, we shall use the following notation: $\varphi \in \mathcal{CAC}(X)$ if and only if $\varphi : X \multimap X$ is an admissible compact absorbing contraction map. Note that the class of admissible compact absorbing contraction maps is quite large. In particular, any compact admissible map is an admissible compact absorbing contraction map. For more information about this class of maps, we refer the reader to [2,13].

By using an argument similar to the one given in the proof of Theorem 42.3 in [13], it is easy to establish the following:

Theorem 5.3. *Let $X \in \text{ANMR}$ and $\varphi \in \mathcal{CAC}(X)$. Then φ is a Lefschetz map and $\Lambda(\varphi) \neq \{0\}$ implies that $\text{Fix}(\varphi) \neq \emptyset$.*

Finally, let us notice that the above theorem can be extended to a class of noncompact maps considered in [14]. Furthermore, the results presented in this section remain valid for multivalued weighted maps. For more details about multivalued weighted maps we recommend: [22,24,25], and the references therein.

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